Traveltime Sensitivity Kernels in Spherical Coordinates: A Progress Report

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Fréchet Kernels

Geometric rays are inadequate for traveltime tomography since for finite frequencies, wave speed variations in a volume around the ray influence arrival times.

The goal in finite frequency tomography is a linear relationship between measured traveltime shifts (relative to some reference) and perturbations in wave speed of the form

$$\tau(\sigma_a, \sigma_b) = \int d^3\mathbf{x}\, \frac{\Delta c(\mathbf{x})}{c_0(\mathbf{x})} K(\mathbf{x}; \sigma_a, \sigma_b),$$

where the function K is known as a Fréchet kernel.

One can calculate kernels relating changes in any medium property to changes in any suitably defined data-derived quantity.

How do we calculate kernel formulas in a consistent, systematic way?

Basic Functional Analysis

The basic tools of functional analysis provide a neat framework for defining and deriving kernels.

A functional is a mapping from a space of functions to a finite-dimensional space of real or complex numbers.

A simple example would be

$$f[g] = \int_a^b dx\, g(x).$$

Functionals can simultaneously be functions of other variables as well, e.g. the above example could be denoted $f[g](a, b)$.

Basic Functional Analysis

Our functional analytic workhorse will be the functional derivative, which is formally defined as

$$\frac{\delta f[g]}{\delta g(a)} = \lim_{\varepsilon \to 0} \frac{f[g(\cdot) + \varepsilon \delta\left(\cdot - a\right)] - f[g(\cdot)]}{\varepsilon}.$$

It may be useful to conceptualize the functional derivative as a gradient in a high dimensional space.

Formulating Kernels

The traveltime shift is a functional of the actual solar model. If we let $\mathbf{q}$ and $\mathbf{q}_0$ denote vectors of functions describing the actual and reference solar models, then we can expand τ in a functional Taylor series about $\mathbf{q}_0$,

$$\tau[\mathbf{q}](\sigma_a,\sigma_b) = \tau[\mathbf{q}_0](\sigma_a,\sigma_b) + \sum_\alpha \int_\odot d^3\mathbf{x}\, \Delta q_\alpha(\mathbf{x}) \left.\frac{\delta\tau}{\delta q_\alpha(\mathbf{x})}(\sigma_a,\sigma_b)\right|_{\mathbf{q}_0}$$
$$+ \frac{1}{2!}\sum_{\alpha,\beta}\int_\odot d^3\mathbf{x}\int_\odot d^3\mathbf{x}'\,\Delta q_\alpha(\mathbf{x})\Delta q_\beta(\mathbf{x}') \left.\frac{\delta^2\tau}{\partial q_\beta(\mathbf{x}')\partial q_\alpha(\mathbf{x})}(\sigma_a,\sigma_b)\right|_{\mathbf{q}_0} + \cdots ,$$

where $\Delta q_\alpha(\mathbf{x}) = q_\alpha(\mathbf{x}) - q_{0,\alpha}(\mathbf{x})$, and σ_a and σ_b are the locations of the observation points on the surface.

Formulating Kernels

Noting that $\tau[\mathbf{q}_0] = 0$, we drop terms of second order and higher in the Δq_α and make the equivalence

$$\tau(\sigma_a, \sigma_b) \equiv \sum_\alpha \int_\odot d^3\mathbf{x}\, \Delta q_\alpha(\mathbf{x})\, \frac{\delta \tau}{\delta q_\alpha(\mathbf{x})}(\sigma_a, \sigma_b)\Big|_{\mathbf{q}_0}.$$

We define K_α, the kernel for the α-th model parameter, as

$$K_\alpha(\mathbf{x}; \sigma_a, \sigma_b) = q_{0,\alpha}(\mathbf{x})\, \frac{\delta \tau}{\delta q_\alpha(\mathbf{x})}(\sigma_a, \sigma_b)\Big|_{\mathbf{q}_0},$$

so that

$$\tau(\sigma_a, \sigma_b) \equiv \sum_\alpha \int_\odot d^3\mathbf{x}\, \frac{\Delta q_\alpha(\mathbf{x})}{q_{0,\alpha}(\mathbf{x})} K_\alpha(\mathbf{x}; \sigma_a, \sigma_b).$$

Formulating Kernels

So what's $\left.\dfrac{\delta\tau}{\delta q_\alpha(\mathbf{x})}(\sigma_a,\sigma_b)\right|_{\mathbf{q}_0}$?

That depends on the precise definition of the traveltime shift.

We will take τ to be the time shift that maximizes the cross-correlation between an actual and reference seismogram.

Calling the cross-correlation function $C(t)$, we expand it in a Taylor series about $t = 0$:

$$C(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} C^{(n)}(0).$$

Formulating Kernels

Assuming that C is a smooth function, we know $C'(\tau) = 0$, so

$$C'(\tau) = \sum_{n=1}^{\infty} \frac{\tau^{n-1}}{(n-1)!} C^{(n-1)}(0) = 0$$

Taking the functional derivative gives us

$$0 = \left. \frac{\delta C'(\tau)}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} = \left. \frac{\delta C'(0)}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} + \sum_{n=2}^{\infty} \left. \left[\frac{\tau^{n-2}}{(n-2)!} C^{(n)}(0) \frac{\delta \tau}{\delta q_\alpha(\mathbf{x})} + \frac{\tau^{n-1}}{(n-1)!} \frac{\delta C^{(n)}(0)}{\delta q_\alpha(\mathbf{x})} \right] \right|_{\mathbf{q}_0} .$$

Since $\tau[\mathbf{q}_0] = 0$, we get

$$0 = \left. \frac{\delta C'(0)}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} + C_0''(0) \left. \frac{\delta \tau}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} ,$$

or

$$\left. \frac{\delta \tau}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} = - \frac{1}{C_0''(0)} \left. \frac{\delta C'(0)}{\delta q_\alpha(\mathbf{x})} \right|_{\mathbf{q}_0} ,$$

where C_0 is the autocorrelation of the synthetic traces for the reference model.

Formulating Kernels

Since C is readily expressible in terms of the physical wavefield, we can certainly evaluate its functional derivative. This leads directly to the first-order Born approximation, which appears in all treatments of Fréchet kernels.

Currently we are developing the code to calculate traveltime kernels, in spherical coordinates, for perturbations in sound speed, density, and fluid velocities relative to model S.

A significant part of this effort is the development of a code to calculate accurate frequency domain Green's functions for model S.

Assumptions

In addition to spherical symmetry, we also make the following assumptions about our reference model:

- Following Birch et al. (2004), we use a source covariance matrix formulation to emulate random excitation:

$$M_{ij}(\sigma_1 - \sigma_2, r_1, r_2, t_1 - t_2) = E\left[S_i(\sigma_1, r_1, t_1) S_j(\sigma_2, r_2, t_2)\right],$$

$$\tilde{M}_{ij}(\Delta\sigma, r_1, r_2, \omega) = \delta_{ir}\delta_{jr}\frac{\delta\left(\theta_1 - \theta_2\right)\delta\left(\phi_1 - \phi_2\right)}{r_1^2 r_2^2 \sin\theta_1}\delta'\left(r_1 - r_s\right)\delta'\left(r_2 - r_s\right)\left|\tilde{f}(\omega)\right|^2.$$

- We presently use a free-surface boundary condition.

Assumptions

- We take a simple damping model that is the product of a function of frequency and a function of radius, a composite of the damping models of Gizon & Birch (2002) and Birch et al. (2004):

$$\Gamma(\omega, r) = g(\omega)h(r),$$

$$g(\omega) = \gamma \left|\frac{\omega}{\omega_*}\right|^{\beta}$$

with $\gamma/2\pi = 100\mu$Hz, $\omega_*/2\pi = 3$mHz, and $\beta = 4.4$, and

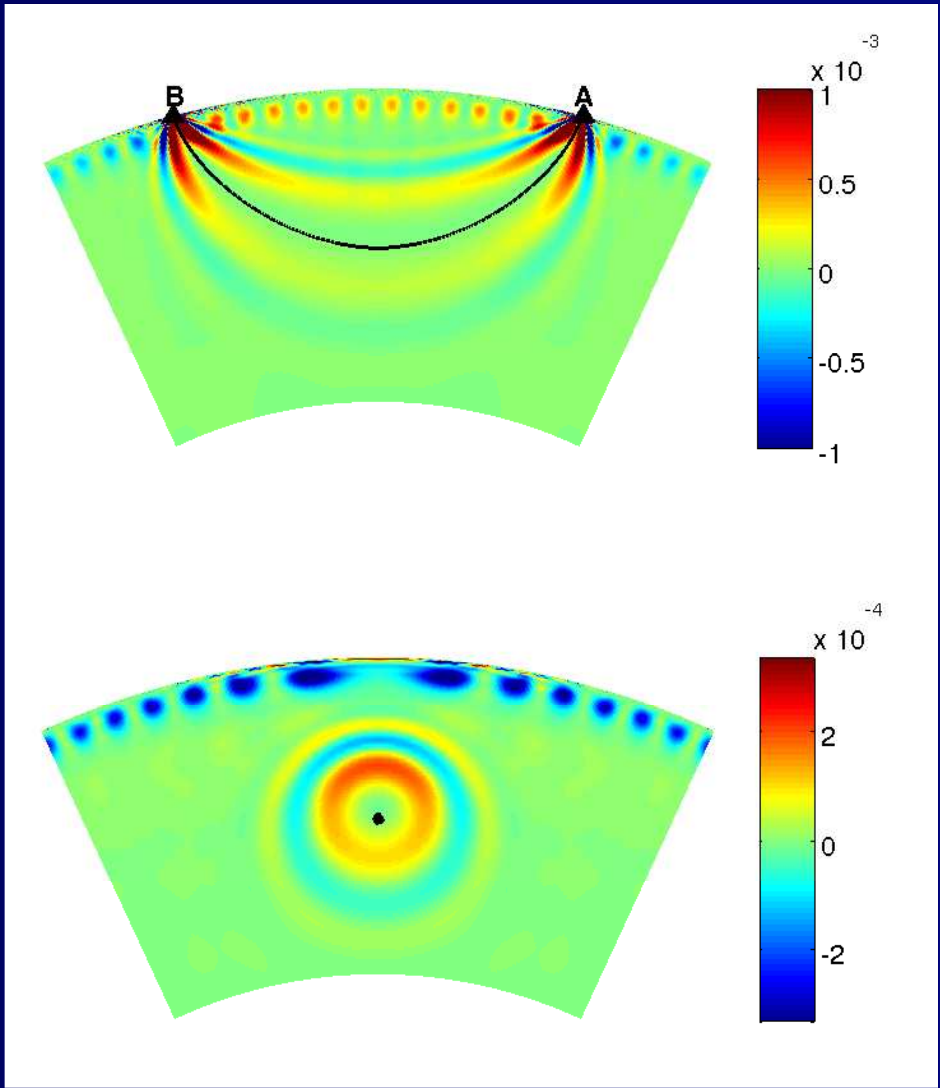
$$h(r) = \exp\left[-\frac{(T(r) - T_c)^2}{(\Delta t)^2}\right]$$

with

$$T(r) = \int_0^r \frac{dr'}{c_0(r')},$$

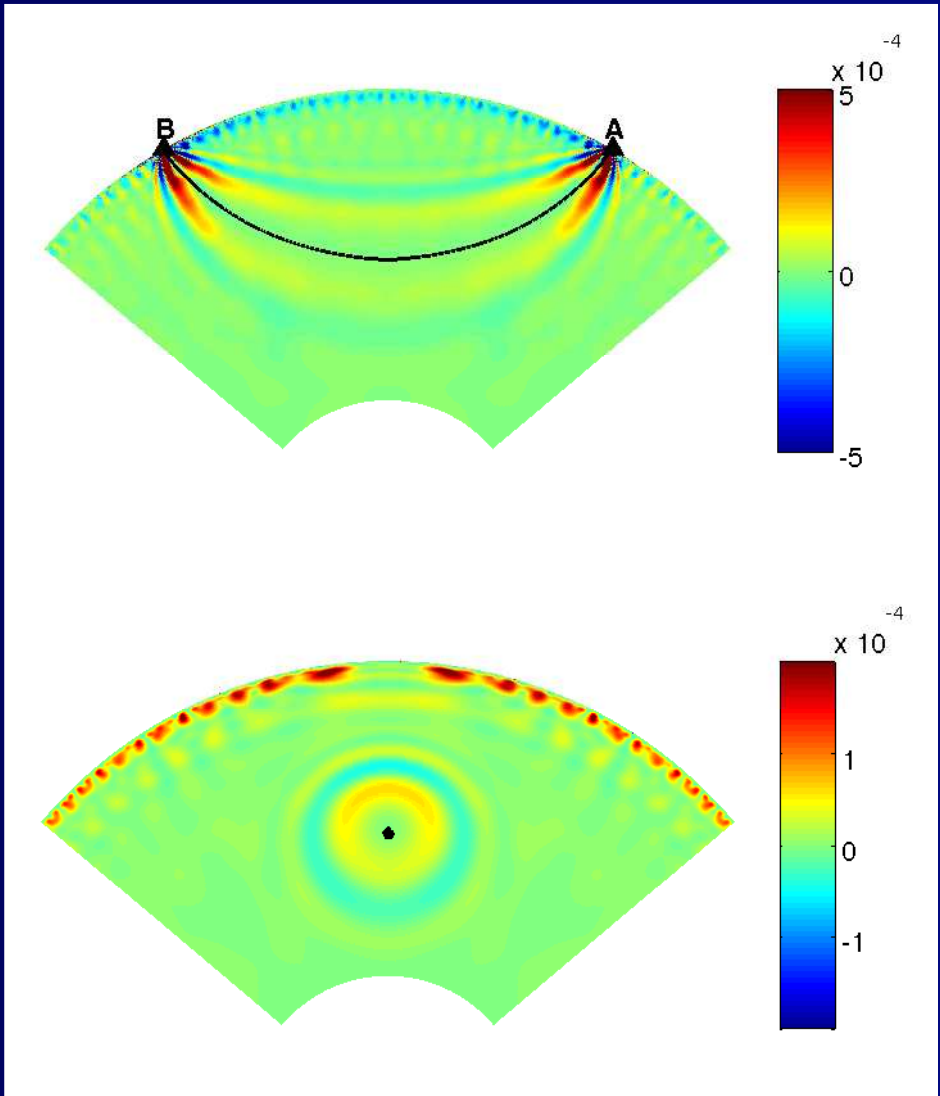
$T_c = T(R)$, where R is the radius of the photosphere, and $\Delta t =$69 s.

Examples



Vertical cross-sections of a sensitivity kernel for squared sound speed for a distance of 30°. The kernel has been scaled by the sound speed and the scale has been severely saturated. (*Top*) The cross-section in the plane of the ray path (black line). (*Bottom*) The cross-section perpendicular to the ray path. The intersection point of the plane and the ray is indicated by the black dot. The total angular range in both sections is 50°, and both extend radially from 0.6 R to the surface. Note that this is the kernel for waves travelling from point B to point A. The units are Mm^{-2}

Examples



Vertical cross-sections of a sensitivity kernel for squared sound speed for a distance of 60°. Again the kernel has been scaled by the sound speed and the scale has been saturated. (*Top*) The cross-section in the plane of the ray path (black line). (*Bottom*) The cross-section perpendicular to the ray path. The intersection point of the plane and the ray is indicated by the black dot. The total angular range in both sections is 100°, and both extend radially from 0.3 R to the surface.

Work in Progress

- Adequately sample small-linewidth modes, probably by accounting for finite time window of observations;
- Incorporate an isothermal atmosphere boundary condition;
- Increase computation speed of kernels by pre-computation of frequently needed quantities;
- Produce volumetric kernels for density, fluid flows, and magnetic fields in addition to wavespeed.