Measurement of the meridional flow from eigenfunction perturbations

Ariane Schad, Markus Roth
Kiepenheuer-Institut für Sonnenphysik

Solar Subsurface Flows from Helioseismology: Problems and Prospects
Helioseismology Workshop
July 21 - 23, 2014, Stanford, CA
I. Theory: Perturbation & coupling of p-modes

> coupling of modes in a „neighborhood“ $K_k$ of a reference mode $k=(n,l,m)$:

$$\xi_k(r, \theta, \phi) = \sum_{k' \in K_k} c_{kk'} \xi_{k'}^0(r, \theta, \phi),$$

perturbed eigenfunction \hspace{1cm} unperturbed eigenfunction

(Lavely & Ritzwoller 1992)

> $c_{kk'}$ – coupling coefficient between mode $k,k'$ ($\approx$coupling strength):

$$c_{kk'} \approx -2 i \frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \int \rho_0 \xi_{k'}^0 \cdot (u \cdot \nabla \xi_k^0) \, d^3r \in \mathbb{C}, \quad k' \in K_k \quad (1. \text{order approximation})$$

I. Theory: Perturbation & coupling of p-modes

Spherical harmonic representation of $u$:

$$ u(r) = \sum_{s=1}^{\infty} \left[ u_s^0(r)Y_s^0(\theta, \phi)e_r + v_s^0(r)\partial_\theta Y_s^0(\theta, \phi)e_\theta \right] $$

radial component  
horizontal component

Conservation of mass:

$$ \rho_0 rs(s+1)v_s^0 = \partial_r (r^2 \rho_0 u_s^0) $$

Polynomial expansion of coupling coefficients:

$$ c_{kk'}(m) = c_{kk'}(m) \approx i \frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \sum_s b_{kk'}^s P_{kk'}^s(m) $$

$b$-coefficients:

$$ b_{kk'}^s = \int_0^R \rho_0(r)K_s^{kk'}(r)u_s^0(r)r^2dr $$


> knowing $b_{kk'}^s$ one can infer the radial flow coefficient $u_s^s(r)$!
II. Effect of mode coupling on global oscillation data

SHT of full-disk Dopplergrams:

\[ o_{lm'}(t) = \int \overline{Y'}_{lm'}(\theta, \phi) W(\theta, \phi) v_D(\theta, \phi, t) d\Omega = \sum_k \alpha_k(t) \sum_{k'' \in K_k} c_{kk''} \xi_{k''}^r(R) L_{k''} \]

\( L_{k'k''} \) - leakage matrix elements (imperfect orthogonality: line of sight projection, solar disk, etc.)

(Fourier) amplitude ratio: between reference mode \( k=(n,l,m) \) and coupling modes \( k' \)

\[ y_{kk'}(\omega_k) = \frac{\tilde{y}_{lm'}(\omega_k)}{\tilde{\omega}_{lm}(\omega_k)} \approx \frac{\sum_{k''} c_{kk''} \xi_{k''}^r(R) L_{k''} L_{k''}}{\sum_{k''} c_{kk''} \xi_{k''}^r(R) L_{k''} L_{k''}} \in \mathbb{C} \]


Estimator: complex gain

\[ G_{lm,l'm}(\omega_{nlm}) = \frac{\langle y_{lm}, l'm(\omega_{nlm}) \rangle}{C S_{lm,l'm}(\omega_{nlm})} = \frac{C S_{lm,l'm}(\omega_{nlm})}{S_{lm}(\omega_{nlm})} \]

(Schad et al., ApJL, 2013)
> given SH time series and leakage matrix

- estimation of amplitude ratios $y_{kk'}$
- estimation of $b$-coefficients $b_{kk'}^s$
- inversion for radial flow coeff. $u_s(r)$
- reconstruction of horizontal flow coefficient $v_s(r)$ from $u_s(r)$

> cross-spectral analysis
> LS – fit
> SOLA inversion method
  (Pijpers & Thompson, 1994)

> polynomial fit

$$
\rho_0 r s (s + 1) v_s^0 = \partial_r (r^2 \rho_0 u_s^0)
$$
IV. Application to MDI data - superimposed flow components (s even)

> MDI data 2004-2010
> s=1,...,8
> complex flow pattern in latitude & depth

How reliable are these results?

(Schad et al., ApJL 2013)
How can we evaluate the method and the reliability of the results?

> comparison of flow measurements from different methods
> forward simulations: e.g. computation of a-ratios for a meridional flow model
> sophisticated simulations of the acoustic wave field (T. Hartlep et al. 2013)

Sources of systematic errors?
> effects of higher order? (approximations in 1. order)
> other large-scale flows, e.g. rotation?
> leakage matrix?
> ...
V. Evaluation: Comparison with subsurface measurements

large-scale flow component (s=2)

small-scale flow component (s=8)


(Schad et al. ApJL, 2013.)

> good agreement at subsurface
V. Evaluation of the method: Simulation study – proof in principle

- flow model: $s=2$, horizontal flow (surface) $\approx 25$ m/s (mid-latitude)
- leakage matrix of MDI
- solar model $S$
- amplitude - ratios + 1% Gaussian noise
- modes with $1 \leq l \leq 200$

> inversion results in good agreement with model
V. Evaluation of the method: Contributions of higher order

\[ \xi_k = \xi_k^0 + \sum_{i \neq k} \left( \frac{H_{ik}}{\omega_k^2 - \omega_i^2} \right) \xi_i^0 + \sum_{i \neq k} \left( \sum_{j \neq k} \frac{H_{ij} H_{jk}}{(\omega_k^2 - \omega_i^2)(\omega_k^2 - \omega_j^2)} \right) \xi_i^0 \]

1. order, imaginary

2. order, real

(with \( H_{ik} \sim c_{ki} \) in first order)

(Schad Dissertation, 2013)

for coupling modes
(n=2, l=120) &
(n'=2, l'=118)

> higher order terms are negligible for the a-ratio
> real part of a-ratio essentially determined by leakage
V. Evaluation of the method: Influence of leakage

Ignore influence of leakage at inversion for the flow:

\[ y_{k,k'}^{(\omega_k)} \approx \frac{\sum_{k''} C_{k,k''} \xi_{k''}^{(R)} L_{k,k''}}{\sum_{k''} C_{k,k''} \xi_{k''}^{(R)} L_{k,k''}} \]

\[ \approx \frac{\xi_{k'}^{(R)} C_{k,k'}}{\xi_k^{(R)} C_{k,k}} \]

\[ L_{k,k''} = L_{k,k} \delta_{k,k''} \]

(Schad Dissertation, 2013)
V. Evaluation of the method: Influence of leakage

Ignore influence of leakage at inversion for the flow:

\[ y_{kk'}(\omega_k) \approx \frac{\sum_{k''} c_{kk''} \xi_{kk''}(R) L_{kk''}}{\sum_{k''} c_{kk''} \xi_{kk''}(R) L_{kk''}} \]

\[ \approx \frac{\xi_{kk'}(R) c_{kk'} \delta_{kk''}}{\xi_{kk}(R) c_{kk}} \]

\[ L_{kk''} = L_{kk} \delta_{kk''} \]

Result:
> s=2 component is redistributed to other harmonic degrees

(Schad Dissertation, 2013)
V. Evaluation of the method: Sources of systematic errors – leakage matrix

Comparison between MDI leakage matrix and leakage estimated from MDI data

Assume no mode coupling due to perturbations (e.g. flows):

$$\frac{L_{lm,n'l'm'}}{L_{l'm',n'l'm'}} \approx \frac{CS_{lm,n'l'm'}(\omega_{n'l'm'})}{S_{l'm'}(\omega_{n'l'm'})}$$

> annual variation of B angle?
> deviation vanishes on average over large times

(Schad Dissertation, 2013)
V. Evaluation of the method: Sources of systematic errors – leakage matrix

\[ L_{lm,n'\nu'm'} = L_{lm,n'\nu'm'}^{(r)} + \beta_{n'\nu'} L_{lm,n'\nu'm'}^{(h)} \]
V. Coupling due to rotation

Toroidal velocity field of solar rotation: (Ritzwoller & Lavely 1991)

\[ \mathbf{u}_{\text{rot}}(r, \theta, \phi) = \Omega(r, \theta) r \sin \theta \mathbf{e}_\theta = - \sum_s w_s(r) \partial_{\theta} Y_s^0(\theta, \phi) \mathbf{e}_\theta \]

Perturbation theory for mode eigenfunctions:

\[ c_{kk'} = c_{kk'}^{(\text{rot})} + c_{kk'}^{(\text{merid.})} \approx \frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \sum_s (a^{s}_{k'k} + i b^{s}_{k'k}) \mathcal{P}^s_{k'k}(m) \]

Expansion coefficients:

\[ a^{s}_{k'k} = \int_0^R \rho_0(r) T_s^{k'k}(r) w_s^0(r) r^2 dr \]
\[ b^{s}_{k'k} = \int_0^R \rho_0(r) K_s^{k'k}(r) w_s^0(r) r^2 dr \]

(Schad Dissertation, 2013)
V. Influence of rotation and meridional flow on a-ratios

Amplitude ratios from MDI data vs. flow model:

> rotation lifts azimuthal symmetry of amplitude ratios

> symmetrization in „m“ can compensate this effect
VI. Restrictions: Frequency resolution and mode frequency separation

Modes should separate in frequency domain (blend into ridges):

> analysis of low and medium degree ($l \leq 200$)

> analysis of higher frequency modes may be affected by bias

> expect bias of about $< 1\%$

Accuracy:

> need of long time series/ high frequency resolution

$$\langle y_{lm,l'm}(\omega_{nlm}) \rangle = \frac{C S_{lm,l'm}(\omega_{nlm})}{S_{lm}(\omega_{nlm})},$$

$$\sigma^2(\omega) = \frac{1}{dof - 2} \frac{S_{l'm}(\omega)}{S_{lm}(\omega)} (1 - \text{Coh}^2(\omega)),$$

dof – degrees of freedom of spectral estimator $\approx$ smoothing kernel width
Summary

- uses full-disk data and exploit spherical geometry
- allows analysis of modes with low and medium degree (deep inversion kernels)
- take into account horizontal and radial component of the flow
- higher order contributions are negligible
- the instruments leakage matrix is important
- what is with the horizontal-to-vertical ratio?
- rotation affects a-ratios - can be compensated
- long time series should be used (frequency resolution, accuracy)