Measurement of the meridional flow from eigenfunction perturbations

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I. Theory: Perturbation & coupling of p-modes

> coupling of modes in a „neighborhood“ K_k of a reference mode k=(n,l,m):

$$\boldsymbol{\xi}_k(r,\theta,\phi) = \sum_{k' \in K_k} c_{kk'} \boldsymbol{\xi}^0_{k'}(r,\theta,\phi)\,,$$

(Lavely & Ritzwoller 1992)

perturbed eigenfunction

unperturbed eigenfunction

> $c_{kk'}$ – coupling coefficient between mode k,k‘ (≈coupling strength):

$$c_{kk'} \approx -2\,\mathrm{i}\,\frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \int \rho_0\, \overline{\boldsymbol{\xi}^0_{k'}} \cdot \underbrace{(\boldsymbol{u} \cdot \nabla \boldsymbol{\xi}^0_k)}_{\text{advection of acoustic wave}}\, d^3\boldsymbol{r} \in \mathbb{C}\,, \quad k' \in K_k$$

(1. order approximation)

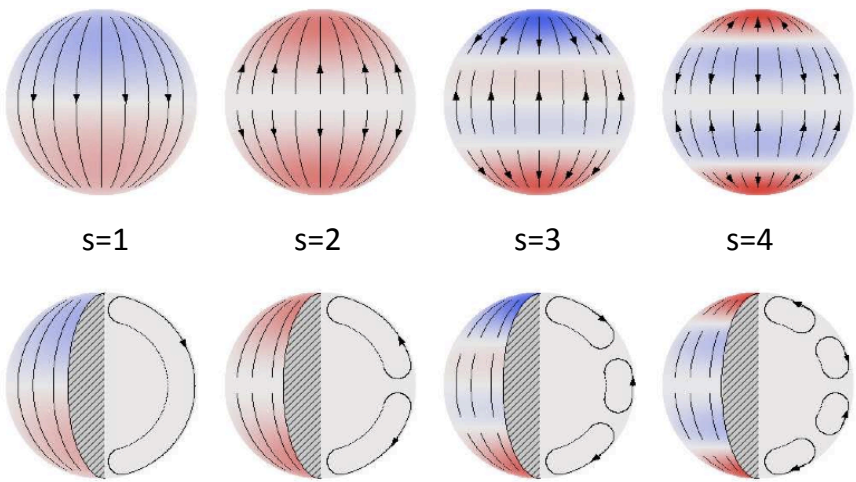
(Schad et al., ApJ, 2011)

I. Theory: Perturbation & coupling of p-modes

Spherical harmonic representation of **u**:

$$\boldsymbol{u}(\boldsymbol{r}) = \sum_{s=1}^{\infty} \left[\underbrace{u_s^0(r) Y_s^0(\theta,\phi)\boldsymbol{e}_r}_{\text{radial component}} + \underbrace{v_s^0(r)\partial_\theta Y_s^0(\theta,\phi)\boldsymbol{e}_\theta}_{\text{horizontal component}} \right]$$

conservation of mass: $\boxed{\rho_0 r s(s+1) v_s^0 = \partial_r (r^2 \rho_0 u_s^0)}$



(Figure source: D. Hathaway, NASA)

Polynomial expansion of coupling coefficients:

$$c_{kk'} = c_{kk'}(m) \approx \mathrm{i} \frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \sum_s b_{k'k}^s \mathcal{P}_{k'k}^s(m)$$

b-coefficients: $\boxed{b_{k'k}^s = \int_0^R \rho_0(r) K_s^{k'k}(r) u_s^0(r) r^2 dr}$

(Schad et al., ApJ, 2011)

> knowing $b^s_{k'k}$ one can infer the radial flow coefficient $u_s(r)$!

II. Effect of mode coupling on global oscillation data

SHT of full-disk Dopplergrams:

$$o_{l'm'}(t) = \int \overline{Y_{l'}^{m'}}(\theta,\phi) W(\theta,\phi) v_D(\theta,\phi,t) d\Omega = \sum_k \alpha_k(t) \sum_{k'' \in K_k} c_{kk''} \xi_{k''}^r(R) L_{k'k''}$$

, $L_{k'k''}$ - leakage matrix elements (imperfect orthogonality: line of sight projection, solar disk, etc.)

(Fourier) amplitude ratio: between reference mode k=(n,l,m) and coupling modes k'

$$y_{kk'}(\omega_k) = \frac{\tilde{o}_{l'm'}(\omega_k)}{\tilde{o}_{lm}(\omega_k)} \approx \frac{\sum_{k''} c_{kk''} \xi_{k''}^r(R) L_{k'k''}}{\sum_{k''} c_{kk''} \xi_{k''}^r(R) L_{kk''}} \in \mathbb{C}$$

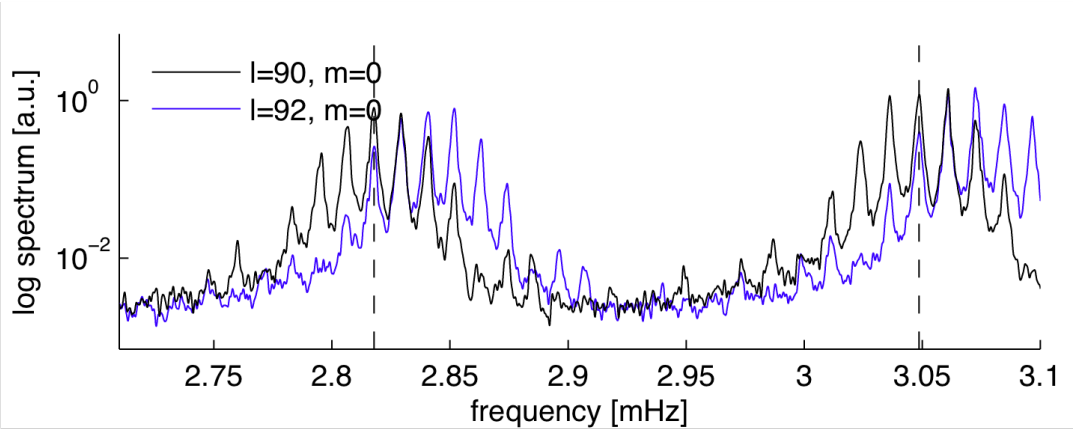
> measure for cross-talk in SHT data

(Schad et al., ApJ, 2011)

Estimator: complex gain

$$G_{lm,l'm}(\omega_{nlm}) := \langle y_{lm,l'm}(\omega_{nlm}) \rangle = \frac{C\, S_{lm,l'm}(\omega_{nlm})}{S_{lm}(\omega_{nlm})},$$

(Schad et al., ApJL, 2013)



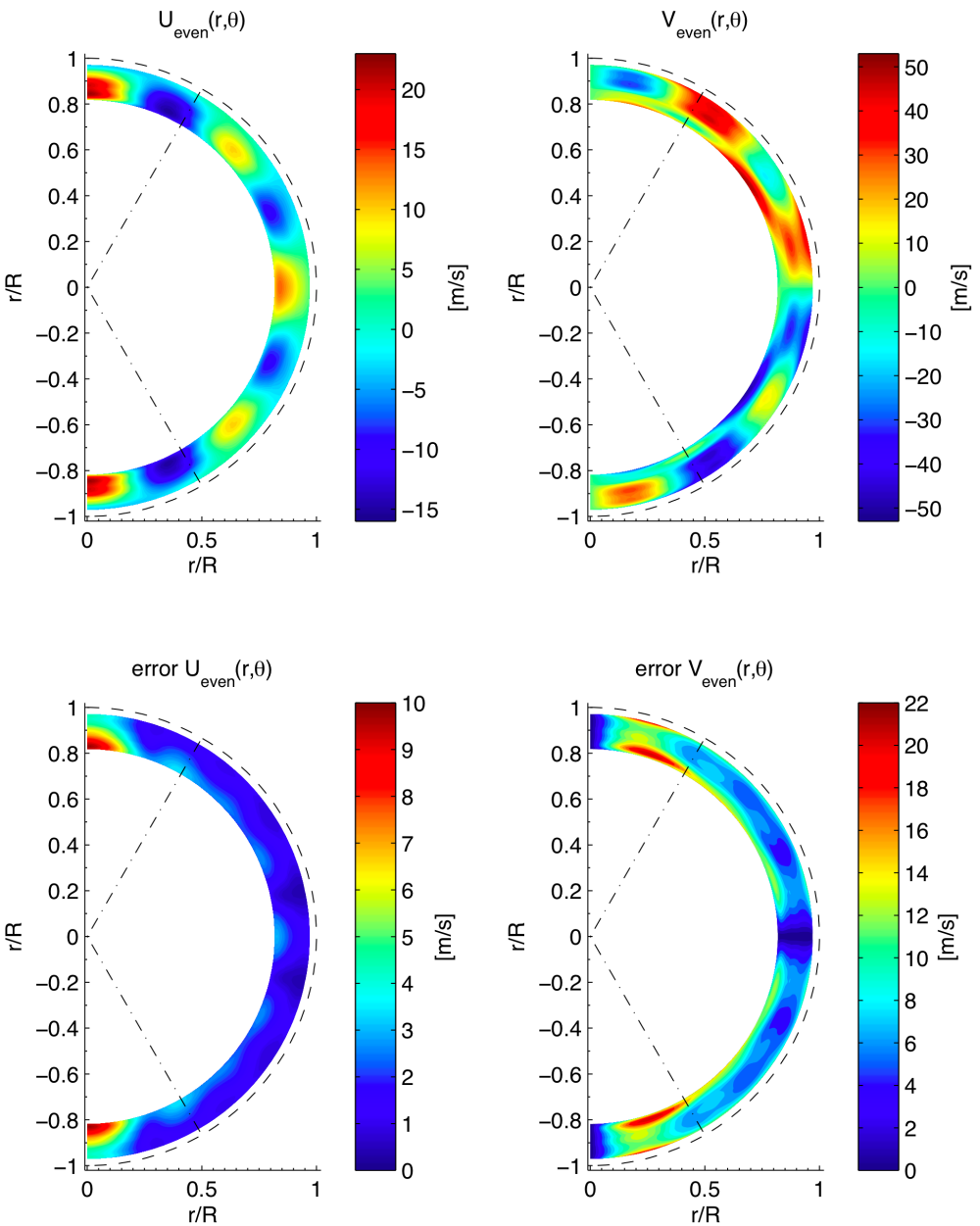
III. Operation scheme

> given SH time series and leakage matrix

- estimation of amplitude ratios $y_{kk'}$ > cross-spectral analysis
- estimation of b-coefficients $b^s_{kk'}$ > LS – fit
- inversion for radial flow coeff. $u_s(r)$ > SOLA inversion method (Pijpers & Thompson, 1994)
- reconstruction of horizontal flow coefficient $v_s(r)$ from $u_s(r)$ > polynomial fit

$$\boxed{\rho_0 r s(s+1) v_s^0 = \partial_r (r^2 \rho_0 u_s^0)}$$

IV. Application to MDI data - superimposed flow components (s even)



> MDI data 2004-2010

> s=1,...,8

> complex flow pattern in latitude & depth

How reliable are these results?

(Schad et al., ApJL 2013)

V. Evaluation of the method - sources of systematic errors

How can we evaluate the method and the reliability of the results?

> comparison of flow measurements from different methods

> forward simulations: e.g. computation of a-ratios for a meridional flow model

> sophisticated simulations of the acoustic wave field (T. Hartlep et al. 2013)

Sources of systematic errors?

> effects of higher order? (approximations in 1. order)

> other large-scale flows, e.g. rotation?

> leakage matrix?

> ...

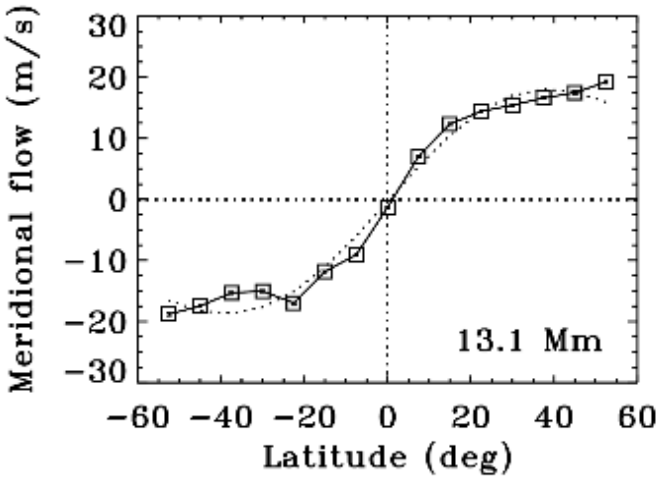
V. Evaluation: Comparison with subsurface measurements

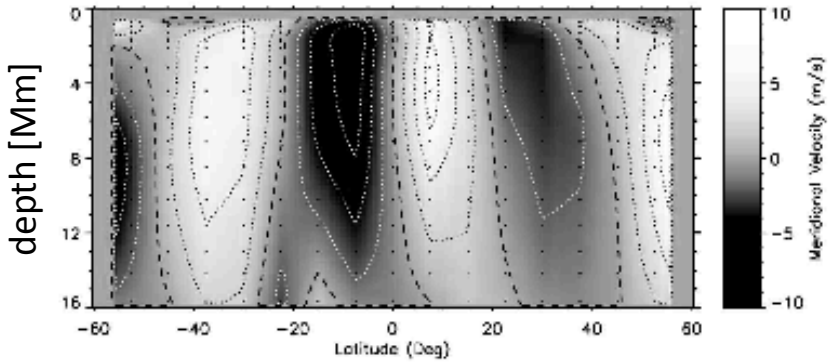
large-scale flow component (s=2)

small-scale flow component (s=8)

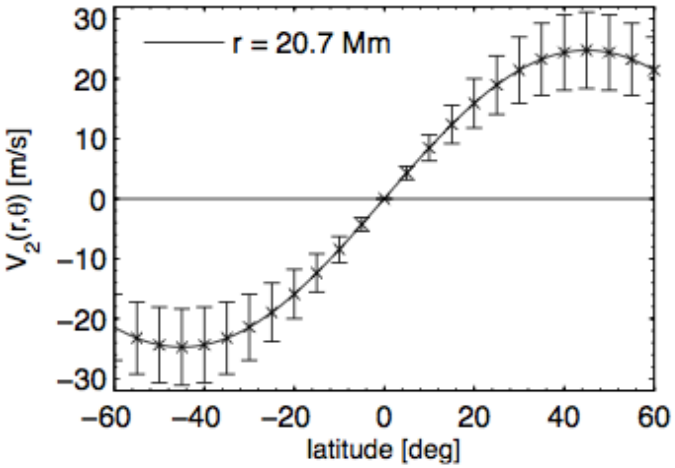
(Komm et al., 2005, ApJ)

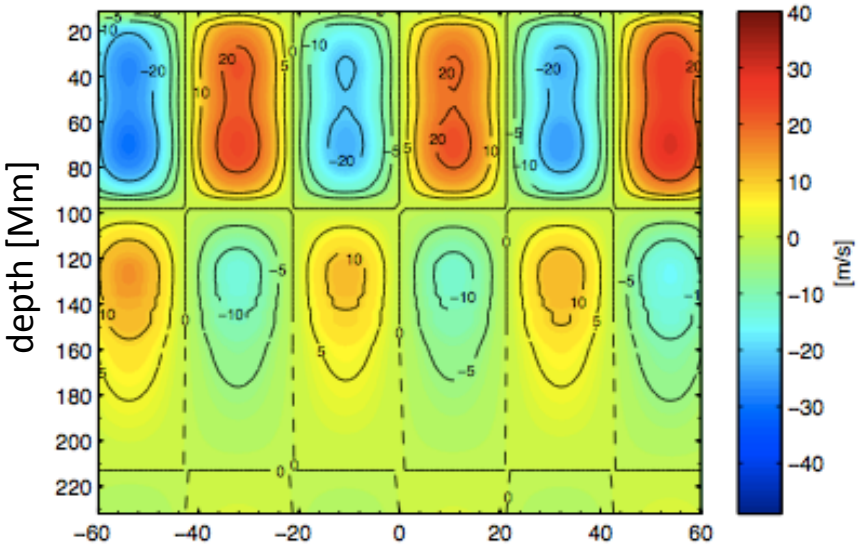
ring-diagram analysis





global inversion method





latitude [deg]

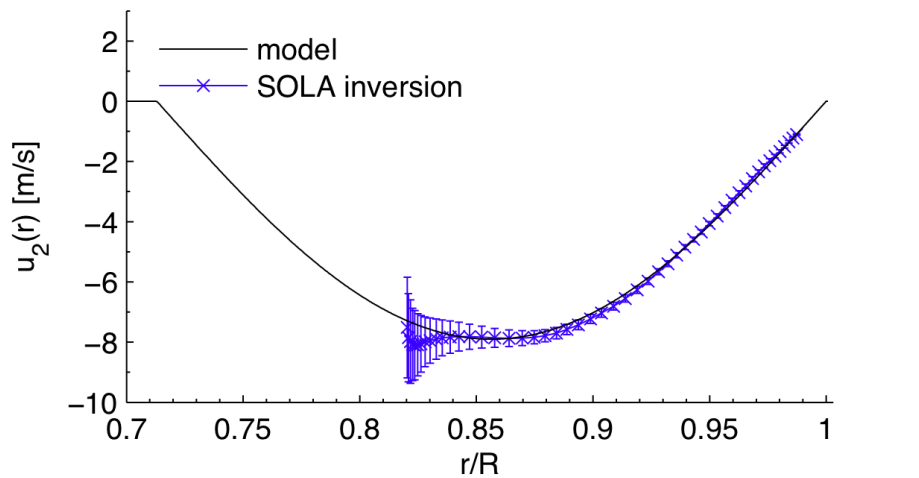
(Schad et al. ApJL, 2013.)

> good agreement at subsurface

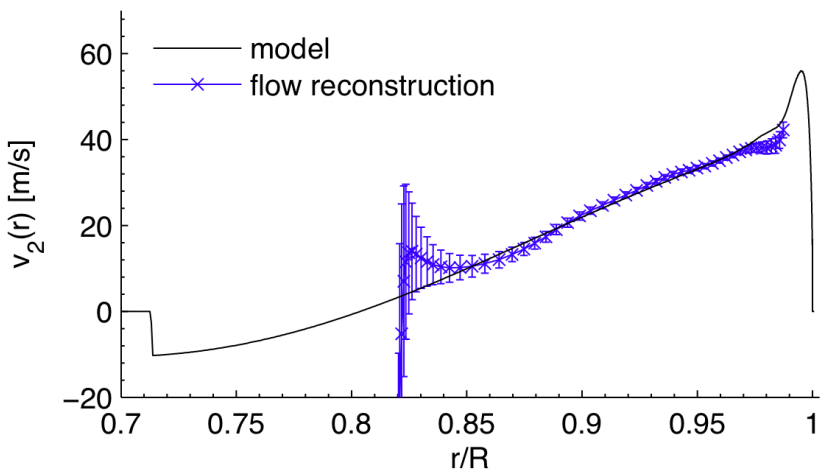
V. Evaluation of the method: Simulation study – proof in principle

- flow model: s=2, horizontal flow (surface) ≈ 25 m/s (mid-latitude)
- leakage matrix of MDI
- solar model S
- amplitude - ratios + 1% Gaussian noise
- modes with 1 ≤ l ≤ 200

radial flow coefficient u_2



horizontal flow coefficient v_2



> inversion results in good agreement with model

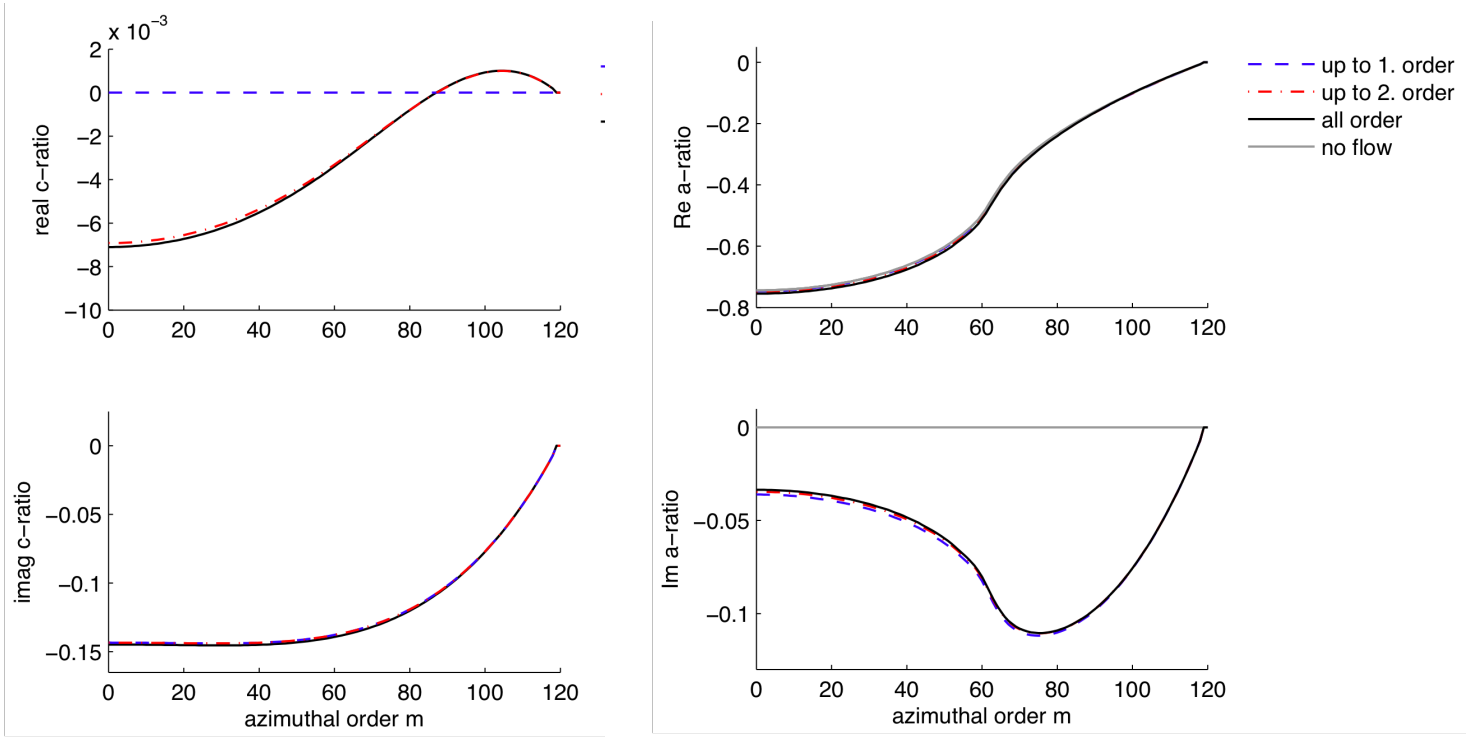
V. Evaluation of the method: Contributions of higher order

$$\xi_k = \xi_k^0 + \sum_{i\neq k} \underbrace{\left(\frac{H_{ik}}{\omega_k^2 - \omega_i^2}\right)}_{\text{1. order, imaginary}} \xi_i^0 + \sum_{i\neq k} \underbrace{\left(\sum_{j\neq k} \frac{H_{ij}\, H_{jk}}{(\omega_k^2 - \omega_i^2)(\omega_k^2 - \omega_j^2)}\right)}_{\text{2. order, real}} \xi_i^0 \,,$$

(with $H_{ik} \sim c_{ki}$ in first order)

(Schad Dissertation, 2013)

for coupling modes
(n=2, l=120) &
(n'=2, l'=118)



> higher order terms are negligible for the a-ratio
> real part of a-ratio essentially determined by leakage

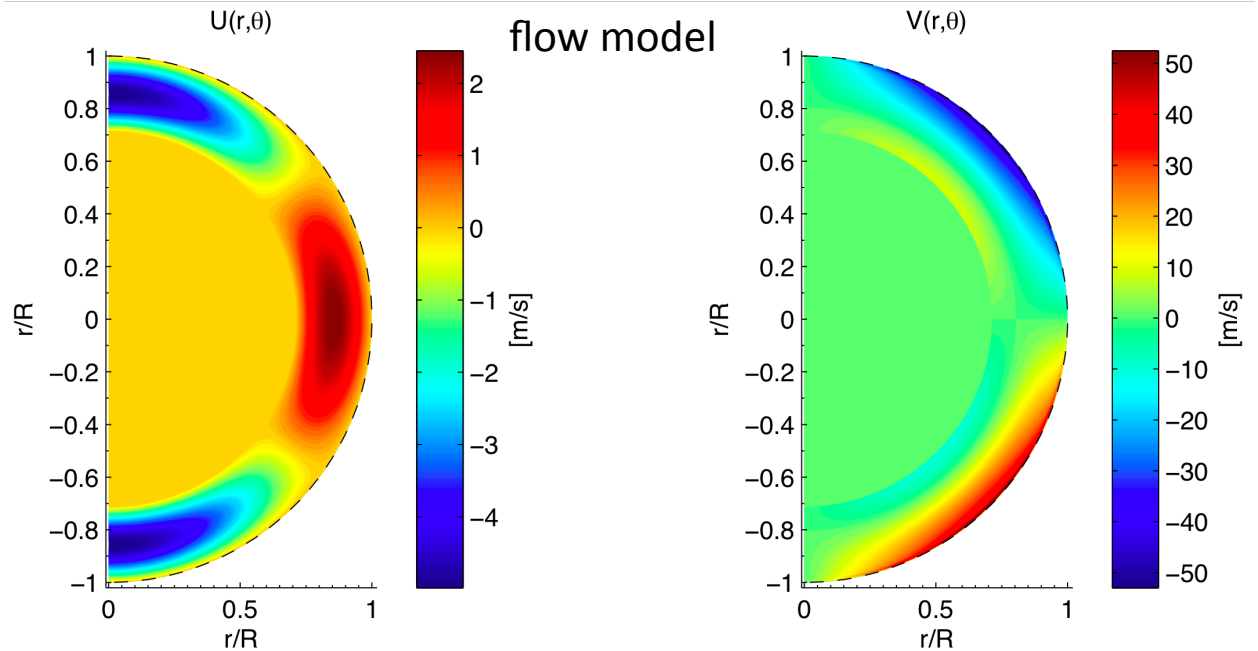
V. Evaluation of the method: Influence of leakage

Ignore influence of leakage at inversion for the flow:

$$y_{kk'}(\omega_k) \approx \frac{\sum_{k''} c_{kk''} \xi^{\tau}_{k''}(R) L_{k'k''}}{\sum_{k''} c_{kk''} \xi^{\tau}_{k''}(R) L_{kk''}}$$

$$\approx \frac{\xi^{\tau}_{k'}(R)}{\xi^{\tau}_{k}(R)} \frac{c_{kk'}}{c_{kk}}$$

$L_{kk''}=L_{kk}\delta_{kk''}$



(Schad Dissertation, 2013)

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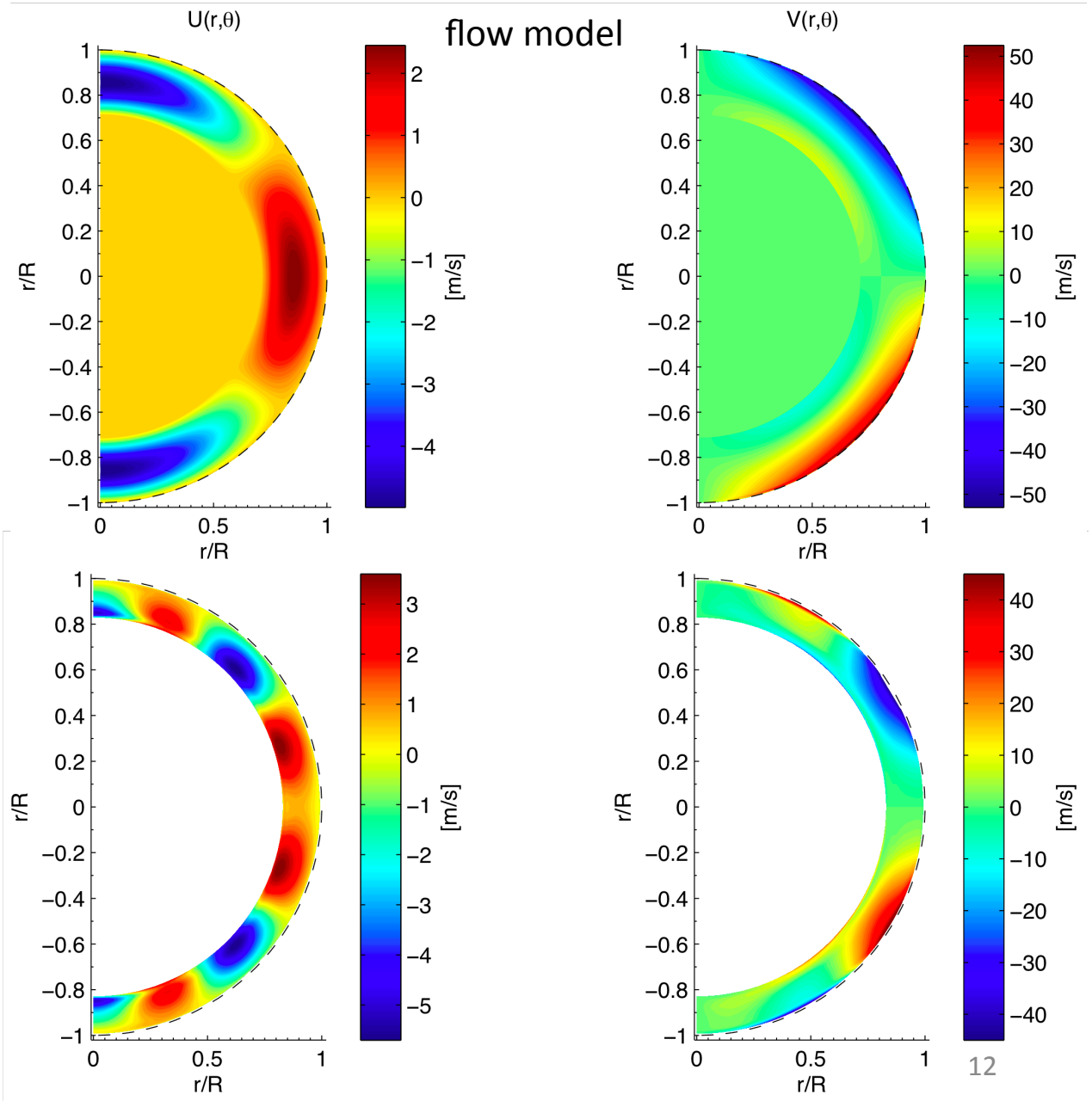
$$\approx \frac{\xi^{\tau}_{k'}(R)}{\xi^{\tau}_{k}(R)} \frac{c_{kk'}}{c_{kk}}$$

$L_{kk''}=L_{kk}\delta_{kk''}$

Result:

> s=2 component is redistributed to other harmonic degrees

(Schad Dissertation, 2013)

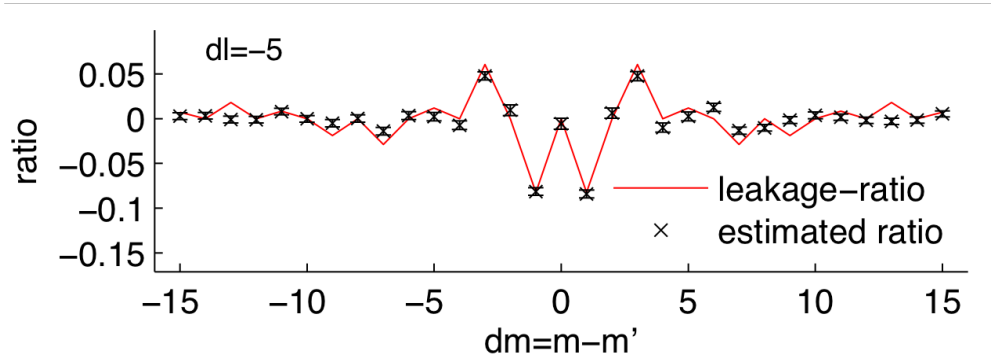


V. Evaluation of the method: Sources of systematic errors – leakage matrix

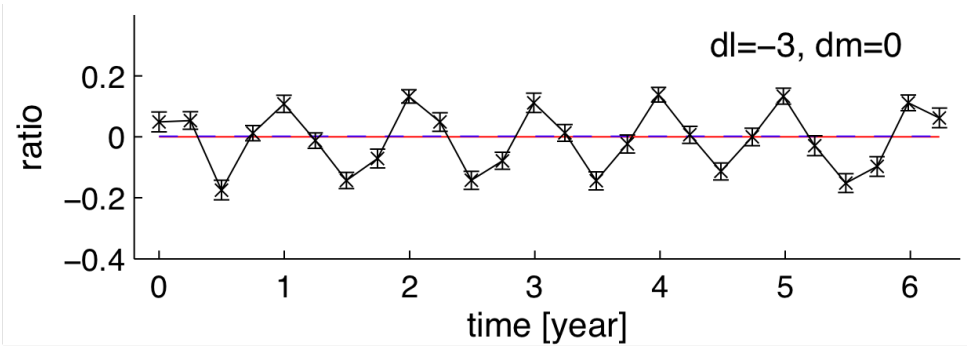
Comparison between MDI leakage matrix and leakage estimated from MDI data

Assume no mode coupling due to perturbations (e.g. flows):

$$\frac{L_{lm,n'l'm'}}{L_{l'm',n'l'm'}} \approx \frac{CS_{lm,l'm'}(\omega_{n'l'm'})}{S_{l'm'}(\omega_{n'l'm'})}$$



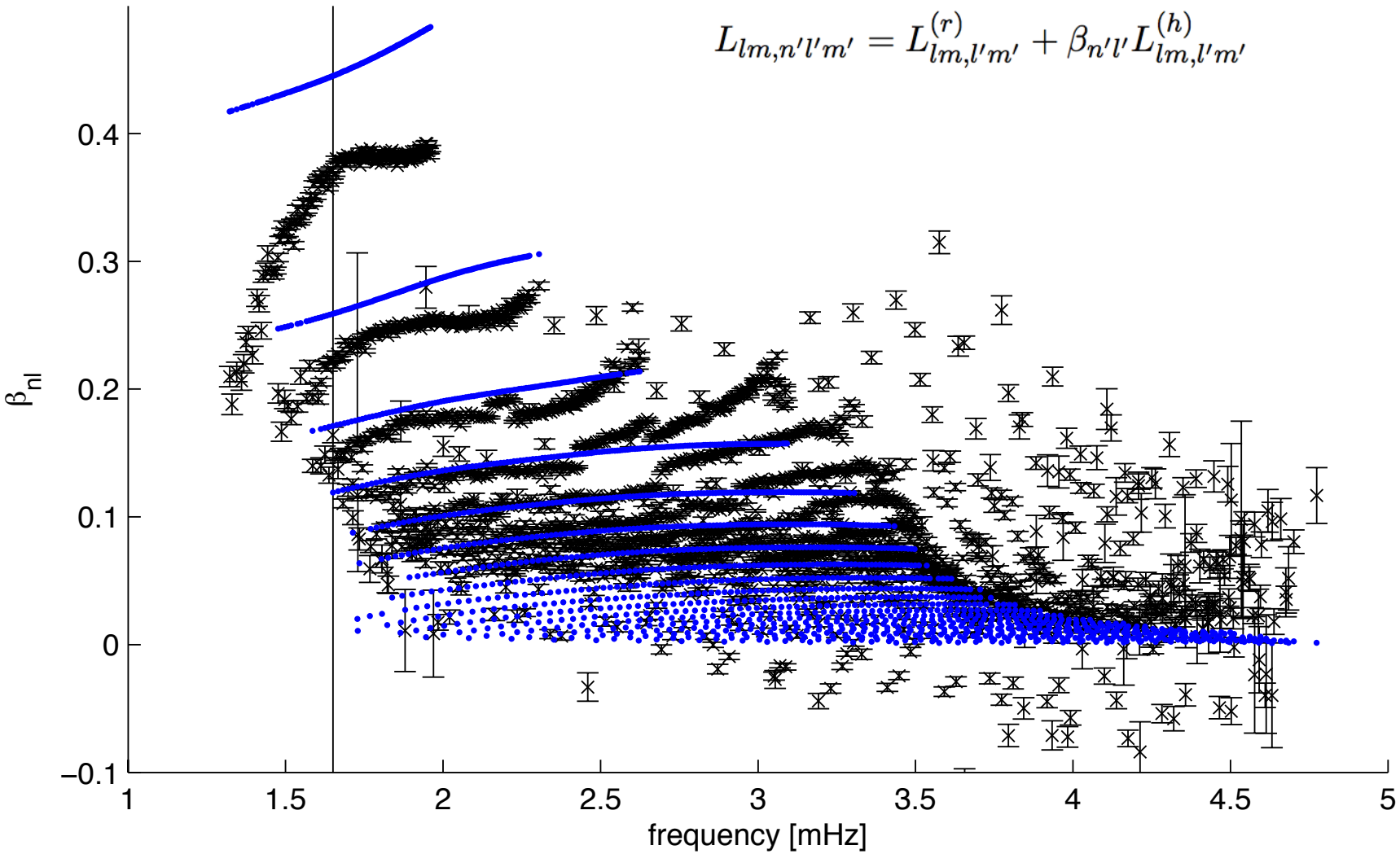
> annual variation of B angle?
> deviation vanishes on average over large times



> for (l'=90, m'=0) and (l=l'+dl, m=m'+dm)

(Schad Dissertation, 2013)

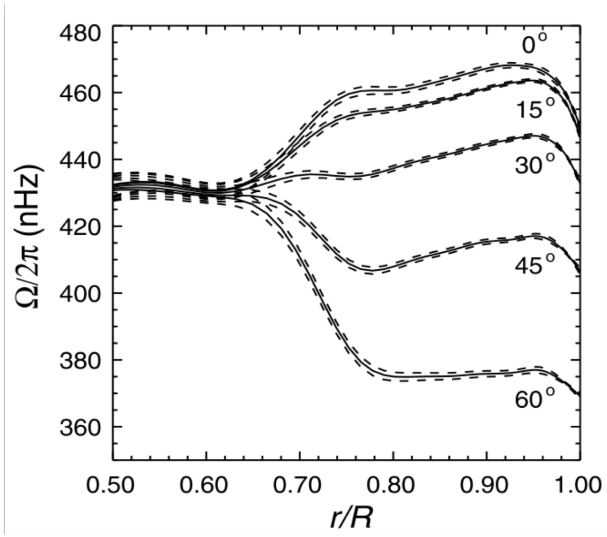
V. Evaluation of the method: Sources of systematic errors – leakage matrix



V. Coupling due to rotation

Toroidal velocity field of solar rotation: (Ritzwoller & Lavely 1991)

$$\boldsymbol{u}_{rot}(r,\theta,\phi) = \Omega(r,\theta) r \sin\theta \, \boldsymbol{e}_\theta = -\sum_s w_s(r) \partial_\theta Y_s^0(\theta,\phi) \boldsymbol{e}_\theta$$



(Howe et al., Sci. 2000)

Perturbation theory for mode eigenfunctions:

$$c_{kk'} = c_{kk'}^{(rot)} + c_{kk'}^{(merid.)} \approx \frac{\omega_k}{\omega_k^2 - \omega_{k'}^2} \sum_s (a_{k'k}^s + \mathrm{i}\, b_{k'k}^s) \mathcal{P}_{k'k}^s(m)$$

Expansion coefficients:

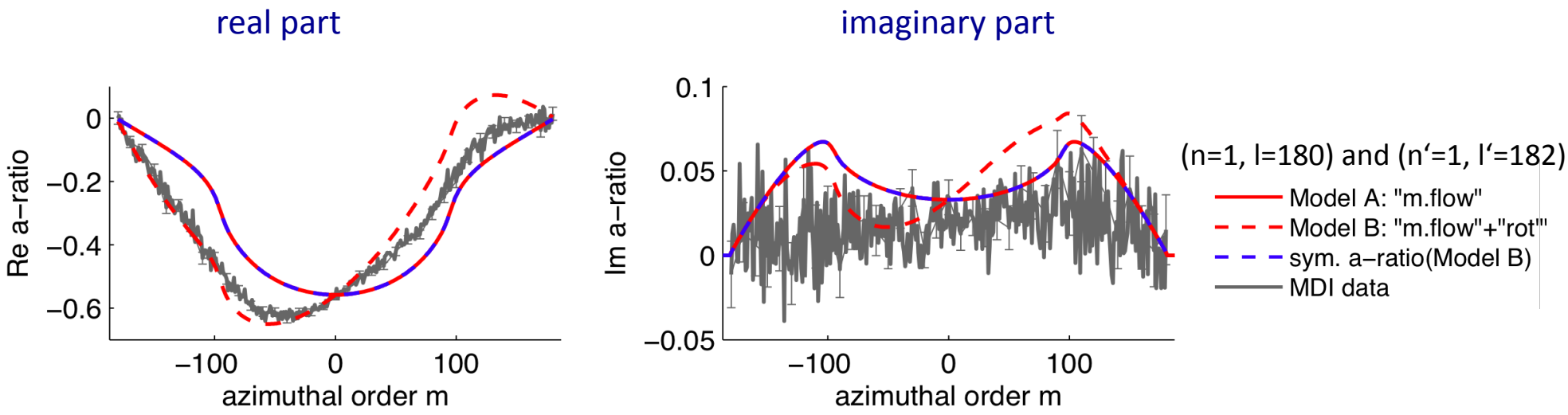
$$a_{k'k}^s = \int_0^R \rho_0(r) T_s^{k'k}(r)\, w_s^0(r)\, r^2 dr$$

$$b_{k'k}^s = \int_0^R \rho_0(r) K_s^{k'k}(r) u_s^0(r) r^2 dr$$

(Schad Dissertation, 2013)

V. Influence of rotation and meridional flow on a-ratios

Amplitude ratios from MDI data vs. flow model:



> rotation lifts azimuthal symmetry of amplitude ratios

> symmetrization in „m“ can compensate this effect

VI. Restrictions: Frequency resolution and mode frequency separation

Modes should separate in frequency domain (blend into ridges):

> analysis of low and medium degree (l ≤ 200)

> analysis of higher frequency modes may be affected by bias

> expect bias of about < 1%

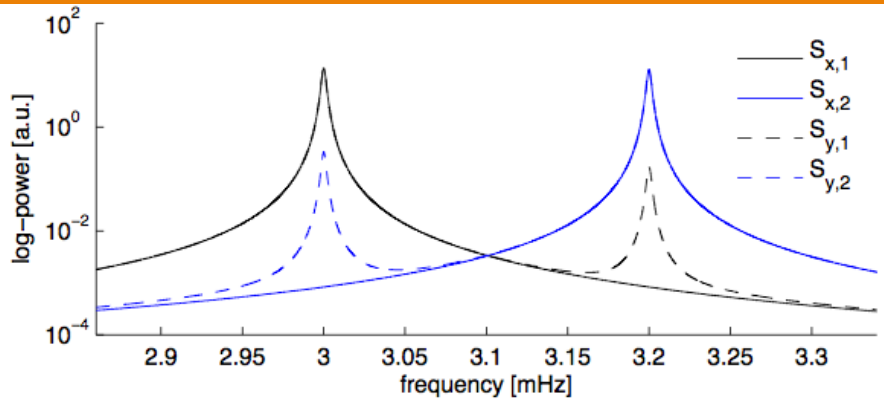
Accuracy:

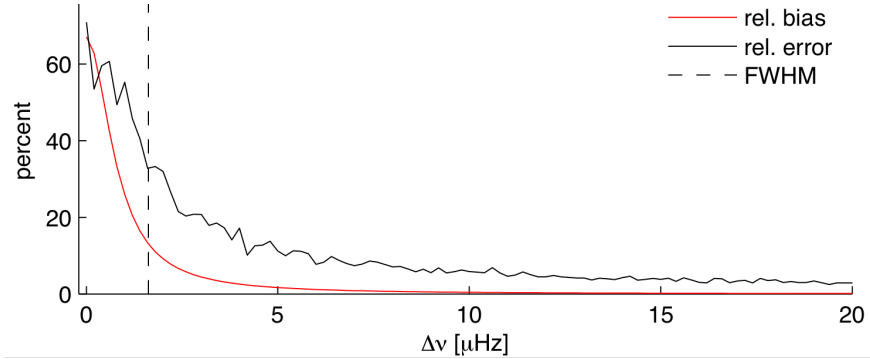
> need of long time series/ high frequency resolution

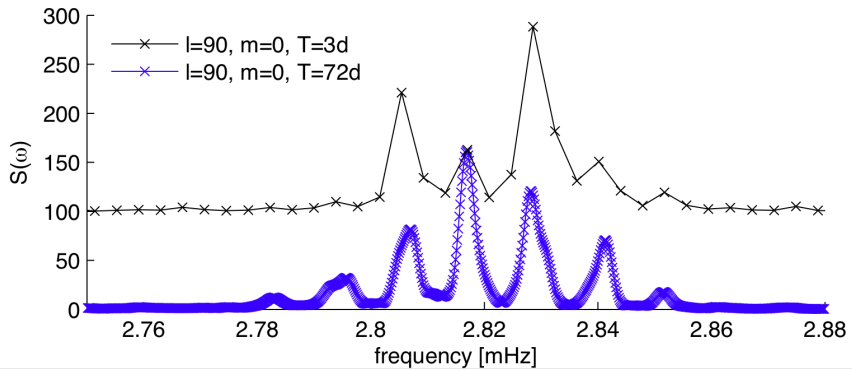
$$\langle y_{lm,l'm}(\omega_{nlm})\rangle = \frac{C\, S_{lm,l'm}(\omega_{nlm})}{S_{lm}(\omega_{nlm})},$$

$$\sigma^2(\omega) = \frac{1}{\mathrm{dof}-2}\frac{S_{l'm}(\omega)}{S_{lm}(\omega)}(1-\mathrm{Coh}^2(\omega)),$$

dof – degrees of freedom of spectral estimator ≈ smoothing kernel width







Summary

- uses full-disk data and exploit spherical geometry
- allows analysis of modes with low and medium degree (deep inversion kernels)
- take into account horizontal and radial component of the flow
- higher order contributions are negligible
- the instruments leakage matrix is important
- what is with the horizontal-to-vertical ratio?
- rotation affects a-ratios - can be compensated
- long time series should be used (frequency resolution, accuracy)